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Hermitian matrices and graphs: singular values and discrepancy

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Abstract

Let $A = (a_{ij})_{i,j=1}^n$ be a Hermitian matrix of size $n \geq 2$, and set

$$\rho(A) = \frac{1}{n^2} \sum_{i,j=1}^n a_{ij},$$

$$\text{disc}(A) = \max_{X,Y \subset [n], X \neq \emptyset, Y \neq \emptyset} \frac{1}{\sqrt{|X||Y|}} \left| \sum_{i \in X} \sum_{j \in Y} (a_{ij} - \rho(A)) \right|.$$

We show that the second singular value $\sigma_2(A)$ of A satisfies

$$\sigma_2(A) \leq C_1 \text{disc}(A) \log n$$

for some absolute constant C_1 , and this is best possible up to a multiplicative constant. Moreover, we construct infinitely many dense regular graphs G such that

$$\sigma_2(A(G)) \geq C_2 \text{disc}(A(G)) \log |G|,$$

where $C_2 > 0$ is an absolute constant and $A(G)$ is the adjacency matrix of G . In particular, these graphs disprove two conjectures of Fan Chung.

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1. Introduction

Given a Hermitian matrix A of size n , let $\mu_1(A) \geq \dots \geq \mu_n(A)$ be its eigenvalues, and $\sigma_1(A) \geq \dots \geq \sigma_n(A)$ be its singular values. As A is Hermitian, the values $\sigma_i(A)$ are the moduli of $\mu_i(A)$ taken in descending order, so $\sigma_1(A)$ is the l_2 operator norm and the spectral radius of A .

Our graph-theoretic notation is standard (e.g., see [1]). For simplicity, graphs are assumed to be defined on the vertex set $[n] = \{1, \dots, n\}$. Occasionally, to remind the reader of this, we write $G(n)$ for a graph of order n , $G(n, m)$ for a graph of order n and size m . We write $e(X)$ for $e(G[X])$ if it is understood which graph G is to be taken. Given a graph G , we let $\mu_i(G) = \mu_i(A(G))$, and $\sigma_i(G) = \sigma_i(A(G))$, where $A(G)$ is the adjacency matrix of G . Given a

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graph G and $X, Y \subset V(G)$, we denote by $e(X, Y)$ the number of the ordered pairs (u, v) such that $u \in X$, $v \in Y$, and u is adjacent to v .

For every graph $G = G(n)$, set $\rho(G) = e(G) \binom{n}{2}^{-1}$ and let

$$\text{disc}_1(G) = \max_{X \subset V(G), X \neq \emptyset} \left\{ \frac{1}{|X|} \left| e(X) - \rho(G) \binom{|X|}{2} \right| \right\},$$

$$\text{disc}_2(G) = \max_{X, Y \subset V(G), X \neq \emptyset, Y \neq \emptyset} \left\{ \frac{1}{\sqrt{|X||Y|}} |e(X, Y) - \rho(G)|X||Y|| \right\}.$$

The function $\text{disc}_1(G)$ is, in fact, Thomason's coefficient α in his definition of (p, α) -jumbled graphs (see, e.g. [11, 12]), on which he based his study of pseudo-random graphs. In principle, $\text{disc}_2(G)$ has the same role as $\text{disc}_1(G)$, although the two invariants may differ significantly for certain graphs, e.g., the star $K_{1,n}$. Chung et al. [8] (see also [10]) used coarser functions to describe the edge distribution of a graph, thus introducing the quasi-random graph properties. Surprisingly, these properties can be expressed in terms of the two largest moduli of the eigenvalues (or equivalently, the two largest singular values) of the adjacency matrix of a graph; we refer the interested reader to [10] for more details. A natural question is, whether similar relations exist between singular values and the functions $\text{disc}_1(G)$ and $\text{disc}_2(G)$.

Chung [5, p. 35] made the following interesting conjecture concerning $\sigma_2(G)$ and $\text{disc}_2(G)$.

Conjecture 1. There is an absolute constant C such that for every regular graph G ,

$$\sigma_2(G) < C \text{disc}_2(G). \quad (1)$$

The main goal of this paper is to study similar questions for graphs and Hermitian matrices. In particular, in Section 2 we define a function $\text{disc}(A)$ for a Hermitian matrix A that naturally extends the function $\text{disc}_2(G)$, and show that there is some constant C' such that for every Hermitian matrix A of size $n \geq 2$, we have

$$\sigma_2(A) < C' \text{disc}(A) \log n. \quad (2)$$

We explicitly construct a nonnegative symmetric matrix showing that (2) is best possible up to a multiplicative constant. Moreover, in Section 3 we construct infinitely many dense regular graphs G such that

$$\mu_2(G) > C'' \text{disc}_2(G) \log |G|$$

for some absolute constant $C'' > 0$, thus disproving Conjecture 1. In fact, as we show in Section 3.4, these graphs disprove also another conjecture of Chung stating a similar problem for Laplacian eigenvalues [6, p. 77].

In particular, in Section 3.1 we show that the bound on $\text{disc}_1(G)$ due to Thomason [13, Theorem 1] can be easily extended to $\text{disc}_2(G)$.

Recently Chung and Graham discussed in [7] quasi-random graph properties of sparse graphs. In particular, they asked whether for sparse graphs small discrepancy implies small second singular value, as in the case of dense graphs. Krivelevich and Sudakov gave an explicit example in [10] that answers this question in the negative. In Section 4 we describe a general construction showing that such examples are not exceptional.

2. Second singular value and discrepancy of Hermitian matrices

Given a matrix $A = (a_{ij})_{i,j=1}^n$ and nonempty sets $I, J \subset [n]$, we denote by $A[I, J]$ the submatrix of the entries a_{ij} with $i \in I$, $j \in J$. We write E_n for the $n \times n$ matrix of all ones, and denote by $\langle \mathbf{x}, \mathbf{y} \rangle$ the standard inner product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. For a Hermitian matrix $A = (a_{ij})_{i,j=1}^n$ set

$$\rho'(A) = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n a_{ij},$$

$$\text{disc}(A) = \max_{X, Y \subset [n], X \neq \emptyset, Y \neq \emptyset} \frac{1}{\sqrt{|X||Y|}} \left| \sum_{i \in X} \sum_{j \in Y} (a_{ij} - \rho'(A)) \right|. \quad (3)$$

In this section we investigate the relationship between $\sigma_2(A)$ and $\text{disc}(A)$. Our main goal is to prove Theorem 2 and to show that the assertion is best possible up to a multiplicative constant.

Observe that for a graph G the value $\rho(G)$ is, generally speaking, different from $\rho'(A(G))$ of the adjacency matrix $A(G)$ of G . However, setting $A = A(G)$, we easily see that for every two nonempty sets $X, Y \subset V(G)$,

$$\begin{aligned} & \frac{1}{\sqrt{|X||Y|}} \left\| \sum_{i \in X} \sum_{j \in Y} (a_{ij} - \rho'(A)) - \sum_{i \in X} \sum_{j \in Y} (a_{ij} - \rho(G)) \right\| \\ & \leq \frac{1}{\sqrt{|X||Y|}} |\rho'(A) - \rho(G)| |X||Y| \\ & = \sqrt{|X||Y|} \left| \frac{1}{n^2} - \frac{1}{n(n-1)} \right| 2e(G) \leq \frac{2e(G)}{n(n-1)}. \end{aligned}$$

Therefore,

$$|\text{disc}_2(G) - \text{disc}(A)| \leq \frac{2e(G)}{n(n-1)} \leq 1, \quad (4)$$

i.e., the function $\text{disc}(A)$ closely approximates $\text{disc}_2(G)$.

2.1. An upper bound on σ_2

Observe that for every Hermitian matrix A of size n , and every $\mathbf{x} \in \mathbb{C}^n$, the Rayleigh principle states that

$$\mu_n(A) \|\mathbf{x}\|^2 \leq \langle A\mathbf{x}, \mathbf{x} \rangle \leq \mu_1(A) \|\mathbf{x}\|^2,$$

thus, we see that for every $\mathbf{x} \in \mathbb{C}^n$,

$$|\langle A\mathbf{x}, \mathbf{x} \rangle| \leq \|A\| \|\mathbf{x}\| = \sigma_1(A) \|\mathbf{x}\|^2. \quad (5)$$

Theorem 2. *There is some constant C such that for every Hermitian matrix A of size $n \geq 2$,*

$$\sigma_2(A) \leq C \text{disc}(A) \log n.$$

Before proceeding to the proof of this theorem let us prove some technical results. We shall prove first a curious lemma that is somewhat stronger than needed.

Lemma 3. *Let $p \geq 1$, $n \geq 1$ and $0 < \varepsilon < 1$. Then for every $\mathbf{x} = (x_i)_1^n \in \mathbb{C}^n$ with $\|\mathbf{x}\|_p = 1$, there is a vector $\mathbf{y} = (y_i)_1^n \in \mathbb{C}^n$ such that y_i take no more than*

$$\left\lceil \frac{8\pi}{\varepsilon} \right\rceil \left\lceil \frac{4}{\varepsilon} \log \frac{4n}{\varepsilon} \right\rceil$$

values and $\|\mathbf{x} - \mathbf{y}\|_p \leq \varepsilon$.

Proof. We shall prove first that if x_i are nonnegative reals, then there is a $\mathbf{y} = (y_i)_1^n$ such that y_i are nonnegative reals, $\|\mathbf{x} - \mathbf{y}\|_p \leq \varepsilon$, and y_i take no more than

$$k = \left\lceil \frac{2}{\varepsilon} \log \frac{2n}{\varepsilon} \right\rceil$$

different values. We may and shall assume $x_1 \geq \dots \geq x_n \geq 0$.

Let us define a sequence $n_1 < n_2 < \dots < n_l \leq n$ as follows. Set $n_1 = 1$; having defined n_i , let s be the maximal index such that

$$x_s \geq \left(1 - \frac{\varepsilon}{2}\right) x_{n_i};$$

if $i = k + 1$ or $s = n$, stop the sequence; otherwise, let $n_{i+1} = s + 1$. Finally, for $1 \leq j \leq n$, set $y_j = x_{n_{i+1}-1}$ if $n_i \leq j < n_{i+1}$ and $y_j = 0$ if $n_k < j \leq n$. For the sake of convenience, set $n_{l+1} = n + 1$. Then, if $l \leq k$,

$$\sum_{j=1}^n |x_j - y_j|^p \leq \sum_{h=1}^l \sum_{j=n_h}^{n_{h+1}-1} \left(\frac{\varepsilon}{2} x_{n_{h+1}-1}\right)^p \leq \left(\frac{\varepsilon}{2}\right)^p \sum_{j=1}^n x_j^p = \left(\frac{\varepsilon}{2}\right)^p < \varepsilon^p.$$

Let now $l \geq k+1$; observe, that the choice of k implies for every $j = n_{k+1}, \dots, n$,

$$x_j \leq \left(1 - \frac{\varepsilon}{2}\right)^k x_1 \leq \left(1 - \frac{\varepsilon}{2}\right)^k \leq \frac{\varepsilon}{2n}.$$

Hence, for $l \geq k+1$, we obtain,

$$\begin{aligned} \sum_{j=1}^n |x_j - y_j|^p &= \sum_{j=1}^{n_{k+1}-1} |x_j - y_j|^p + \sum_{j=n_{k+1}}^n x_j^p \\ &\leq \sum_{h=1}^k \sum_{j=n_h}^{n_{h+1}-1} \left(\frac{\varepsilon}{2} x_{n_{h+1}-1}\right)^p + n \left(\frac{\varepsilon}{2n}\right)^p \leq \left(\frac{\varepsilon}{2}\right)^p + \left(\frac{\varepsilon}{2}\right)^p \leq \varepsilon^p. \end{aligned}$$

Consequently, $\|\mathbf{x} - \mathbf{y}\|_p \leq \varepsilon$, as required.

Let now $\mathbf{x} = (x_j)_1^n \in \mathbb{C}^n$ be an arbitrary vector and for every $j \in [n]$, let

$$x_j = |x_j| \exp(\theta_j 2\pi i),$$

where $0 \leq \theta_j < 1$. Set

$$\mathbf{x}^* = (|x_1|, \dots, |x_n|).$$

According to the above, there exists $\mathbf{z} = (z_j)_1^n$, such that $\|\mathbf{x}^* - \mathbf{z}\|_p \leq \varepsilon/2$, $z_j \geq 0$ and z_j take at most

$$k = \left\lceil \frac{4}{\varepsilon} \log \frac{4n}{\varepsilon} \right\rceil$$

different values. Let

$$m = \left\lceil \frac{8\pi}{\varepsilon} \right\rceil.$$

We shall show that the vector $\mathbf{y} = (y_i)_1^n$ defined by

$$|y_j| = z_j, \quad \arg(y_j) = \frac{\lfloor m\theta_j \rfloor}{m} 2\pi$$

is as required. Let us first check that $\|\mathbf{x} - \mathbf{y}\|_p \leq \varepsilon$. Indeed, define $\mathbf{z}^* = (z_j^*)_1^n$ by

$$z_j^* = z_j \exp(\theta_j 2\pi i).$$

We have, by the triangle inequality,

$$\|\mathbf{x} - \mathbf{y}\|_p \leq \|\mathbf{x} - \mathbf{z}^*\|_p + \|\mathbf{y} - \mathbf{z}^*\|_p. \quad (6)$$

On the one hand,

$$\sum_{j=1}^n |x_j - z_j^*|^p = \sum_{j=1}^n |x_j| - z_j|^p \exp(\theta_j 2\pi i)^p = \sum_{j=1}^n |x_j| - z_j|^p \leq \left(\frac{\varepsilon}{2}\right)^p. \quad (7)$$

On the other hand,

$$\left| \exp\left(\frac{\lfloor m\theta_j \rfloor}{m} 2\pi i\right) - \exp(\theta_j 2\pi i) \right| \leq 2 \sin\left(\frac{1}{2m} 2\pi\right) < \frac{2\pi}{m} \leq \frac{\varepsilon}{4},$$

and hence,

$$\begin{aligned} \sum_{j=1}^n |y_j - z_j^*|^p &\leq \sum_{j=1}^n \left| z_j \exp(\theta_j 2\pi i) - z_j \arg\left(\frac{\lfloor m\theta_j \rfloor}{m} 2\pi i\right) \right|^p \leq \sum_{j=1}^n \left(\frac{\varepsilon}{4}\right)^p |z_j|^p \\ &= \varepsilon^p (\|\mathbf{z}\|_p)^p \leq \left(\frac{\varepsilon}{4}\right)^p \left(\frac{\varepsilon}{2} + \|\mathbf{x}\|_p\right)^p < \left(\frac{\varepsilon}{2}\right)^p. \end{aligned}$$

Hence, in view of (6) and (7), we obtain

$$\|\mathbf{x} - \mathbf{y}\|_p \leq \left(\left(\frac{\varepsilon}{2}\right)^p + \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p} \leq \varepsilon.$$

To complete the proof, observe that y_i take at most

$$km \leq \left\lceil \frac{8\pi}{\varepsilon} \right\rceil \left\lceil \frac{4}{\varepsilon} \log \frac{4n}{\varepsilon} \right\rceil$$

different values. \square

We say that a partition $X = \bigcup_{i=1}^m P_i$ is *proper* if the sets P_i are nonempty.

Lemma 4. Let $B = (b_{ij})_{i,j=1}^n$ be a Hermitian matrix and $[n] = \bigcup_{i=1}^m P_i$ be a proper partition. Let $\mathbf{y} \in \mathbb{C}^m$, and $\mathbf{x} = (x_i)_{i=1}^n \in \mathbb{C}^n$ be such that $x_i = y_j$ for every $i \in P_j$. Then the Hermitian matrix $C = (c_{ij})_{i,j=1}^m$ defined by

$$c_{ij} = \frac{1}{\sqrt{|P_i||P_j|}} \sum_{r \in P_i} \sum_{s \in P_j} b_{rs}$$

satisfies

$$|\langle B\mathbf{x}, \mathbf{x} \rangle| \leq \sigma_1(C) \|\mathbf{x}\|^2.$$

Proof. For every $k \in [m]$, set $t_k = \sqrt{|P_k|} y_k$, and let $\mathbf{t} = (t_1, \dots, t_m)$, so that $\|\mathbf{t}\| = \|\mathbf{x}\|$. Also, we see that

$$\begin{aligned} \langle B\mathbf{x}, \mathbf{x} \rangle &= \sum_{i=1}^n \sum_{j=1}^n b_{ij} x_i \bar{x}_j = \sum_{i=1}^m \sum_{j=1}^m \frac{t_i \bar{t}_j}{\sqrt{|P_i||P_j|}} \sum_{r \in P_i} \sum_{s \in P_j} b_{rs} \\ &= \sum_{i=1}^m \sum_{j=1}^m c_{ij} t_i \bar{t}_j = \langle C\mathbf{t}, \mathbf{t} \rangle. \end{aligned}$$

Hence, from (5), we obtain

$$|\langle B\mathbf{x}, \mathbf{x} \rangle| \leq \sigma_1(C) \|\mathbf{t}\|^2 = \sigma_1(C) \|\mathbf{x}\|^2,$$

completing the proof. \square

Proof of Theorem 2. Set $\rho' = \rho'(A)$ and let

$$B = A - \rho' E_n.$$

Our first goal is to show that

$$\sigma_2(A) \leq \sigma_1(B). \quad (8)$$

Indeed, we have

$$\begin{aligned} \mu_1(A - B) &= \mu_1(\rho' E_n) = \rho' n, \\ \mu_k(A - B) &= \mu_k(\rho' E_n) = 0 \quad \text{for } k = 2, \dots, n. \end{aligned}$$

Weyl's inequalities (e.g., see [9, p. 181]) imply, that if C and D are two Hermitian matrices of order n then

$$\mu_2(C + D) \leq \mu_2(C) + \mu_1(D)$$

and

$$\mu_n(C + D) \leq \mu_n(C) + \mu_1(D).$$

Hence, we see that

$$\mu_2(A) \leq \mu_1(B) + \mu_2(A - B) = \mu_1(B) + \mu_2(\rho E_n) = \mu_1(B)$$

and thus,

$$\mu_2(A) \leq \mu_1(B) \leq \sigma_1(B). \quad (9)$$

Similarly,

$$\mu_1(-B) + \mu_n(A) \geq \mu_n(A - B) = \mu_n(\rho E_n) = 0$$

and thus,

$$\sigma_1(B) \geq \mu_1(-B) \geq -\mu_n(A).$$

This, together with (9), implies (8).

Let now $\mathbf{x} \in \mathbb{C}^n$ be a unit vector such that $|\langle B\mathbf{x}, \mathbf{x} \rangle| = \sigma_1(B)$. Applying Lemma 3 with $\varepsilon = \frac{1}{3}$, we can find a vector $\mathbf{y} = (y_i)_{i=1}^n \in \mathbb{C}^n$ satisfying

$$\|\mathbf{x} - \mathbf{y}\| \leq \frac{1}{3},$$

such that y_i take m distinct values $\alpha_1 < \dots < \alpha_m$, where

$$m \leq \left\lceil \frac{8\pi}{1/3} \right\rceil \left\lceil \frac{4}{1/3} \log \frac{4n}{1/3} \right\rceil.$$

For every $i \in [m]$, let

$$P_i = \{j : y_j = \alpha_i\};$$

clearly, $[n] = P_1 \cup \dots \cup P_m$ is a proper partition.

We shall prove that

$$\sigma_1(B) \leq \frac{9}{2} |\langle B\mathbf{y}, \mathbf{y} \rangle|. \quad (10)$$

Indeed, we have

$$\begin{aligned} \langle B\mathbf{x}, \mathbf{x} \rangle - \langle B\mathbf{y}, \mathbf{y} \rangle &= \langle B(\mathbf{y} - \mathbf{x}), \mathbf{x} \rangle + \overline{\langle \mathbf{y}, B(\mathbf{x} - \mathbf{y}) \rangle} \\ &\leq (\|\mathbf{y}\| + \|\mathbf{x}\|) \|B(\mathbf{y} - \mathbf{x})\| \\ &\leq \left(2 + \frac{1}{3}\right) \sigma_1(B) \|\mathbf{x} - \mathbf{y}\| \leq \frac{7}{9} \sigma_1(B). \end{aligned}$$

Hence if $\langle B\mathbf{x}, \mathbf{x} \rangle = \sigma_1(B)$ then (10) holds. Also, we have

$$\begin{aligned} \langle B(\mathbf{x} - \mathbf{y}), \mathbf{x} \rangle + \overline{\langle \mathbf{y}, B(\mathbf{x} - \mathbf{y}) \rangle} &\geq -(\|\mathbf{y}\| + \|\mathbf{x}\|) \|B(\mathbf{y} - \mathbf{x})\| \\ &\geq -\left(2 + \frac{1}{3}\right) \sigma_1(B) \|\mathbf{x} - \mathbf{y}\| \geq -\frac{7}{9} \sigma_1(B) \end{aligned}$$

and thus, (10) holds also if $\langle B\mathbf{x}, \mathbf{x} \rangle = -\sigma_1(B)$.

Define the Hermitian matrix $C = (c_{ij})_{i,j=1}^m$ by

$$c_{ij} = \frac{1}{\sqrt{|P_i||P_j|}} \sum_{r \in P_i} \sum_{s \in P_j} b_{rs}.$$

Applying Lemma 4 to the partition $[n] = P_1 \cup \dots \cup P_m$ and the vector \mathbf{y} we find that

$$|\langle B\mathbf{y}, \mathbf{y} \rangle| \leq \sigma_1(C) \|\mathbf{y}\|^2 \leq \sigma_1(C) \left(\|\mathbf{x}\| + \frac{1}{3} \right)^2 = \frac{16}{9} \sigma_1(C).$$

Hence, in view of (8) and (10), we see that

$$\sigma_2(A) \leq \sigma_1(B) \leq \frac{9}{2} \sigma_1(C).$$

Observe that,

$$\begin{aligned} \sigma_1(C) &= \max(|\mu_1(C)|, |\mu_n(C)|) \leq m \max_{i,j \in [m]} |c_{ij}| \\ &\leq \left\lceil \frac{8\pi}{1/3} \right\rceil \left\lceil \frac{4}{1/3} \log \frac{4n}{1/3} \right\rceil \max_{i,j \in [m]} |c_{ij}|. \end{aligned}$$

Since,

$$\max_{i,j \in [m]} |c_{ij}| \leq \text{disc}(A),$$

we obtain

$$\sigma_2(A) \leq 8 \left\lceil \frac{8\pi}{1/3} \right\rceil \left\lceil \frac{4}{1/3} \log \frac{4n}{1/3} \right\rceil \text{disc}(A), \quad (11)$$

and the proof is completed. \square

In the arguments above we made no attempt to optimize the constant in Theorem 2. As the right-hand side of (11) is bounded above by

$$(7296 \log n + 18240) \text{disc}(A),$$

we can take C to be 33612.

2.2. Tightness of the upper bound on σ_2

For $n = 2k \geq 2$, let $A' = (a'_{ij})_{i,j=1}^k$ be defined by

$$a'_{ij} = \frac{1}{\sqrt{ij}}$$

and let $A = (a_{ij})_{i,j=1}^n$ be the block matrix

$$A = \begin{pmatrix} E_k + A' & E_k - A' \\ E_k - A' & E_k + A' \end{pmatrix}.$$

Clearly, A is nonnegative and symmetric. As we shall see the matrix A shows that Theorem 2 is best possible up to a multiplicative constant.

Theorem 5. *For the matrix A defined above we have*

$$\mu_2(A) \geq \frac{1}{2} \operatorname{disc}(A) \log n. \quad (12)$$

Proof. In fact, we shall show that $\mu_2(A)$ and $\operatorname{disc}(A)$ satisfy

$$\mu_2(A) \geq 2 \log n$$

and

$$\operatorname{disc}(A) < 4. \quad (13)$$

Indeed, the sum of every row of A is exactly n , and, since A is nonnegative, it follows that $\mu_1(A) = n$. Note that the vector $\mathbf{j} \in \mathbb{R}^n$ of all ones is an eigenvector of A to $\mu_1(A)$. By the Rayleigh principle

$$\mu_2(A) = \max_{\mathbf{y} \perp \mathbf{j}, \mathbf{y} \neq \mathbf{0}} \frac{\langle A\mathbf{y}, \mathbf{y} \rangle}{\|\mathbf{y}\|^2},$$

so our goal is to find a nonzero $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} \perp \mathbf{j}$, and the ratio $\langle A\mathbf{y}, \mathbf{y} \rangle / \|\mathbf{y}\|^2$ is sufficiently large.

Define the vector $\mathbf{y} = (y_i)_{i=1}^n$ by

$$y_i = \begin{cases} 1/\sqrt{i} & \text{if } i \leq k, \\ -1/\sqrt{i-k} & \text{if } i > k. \end{cases}$$

From

$$\sum_{i=1}^{2k} y_i = \sum_{i=1}^k \frac{1}{\sqrt{i}} - \sum_{i=k+1}^{2k} \frac{1}{\sqrt{i-k}} = 0$$

we see that $\mathbf{y} \perp \mathbf{j}$. Setting

$$\zeta_k = \sum_{i=1}^k \frac{1}{i},$$

we deduce

$$\|\mathbf{y}\|^2 = \sum_{i=1}^k \frac{1}{i} + \sum_{i=k+1}^n \frac{1}{i-k} = 2 \sum_{i=1}^k \frac{1}{i} = 2\zeta_k.$$

Next, we shall compute $\langle A\mathbf{y}, \mathbf{y} \rangle$. Recall that

$$a_{ij} = \begin{cases} 1 + 1/\sqrt{ij} & \text{if } i \leq k, j \leq k, \\ 1 - 1/\sqrt{ij} & \text{if } i \leq k, j > k, \\ 1 - 1/\sqrt{ij} & \text{if } i > k, j \leq k, \\ 1 + 1/\sqrt{ij} & \text{if } i > k, j > k. \end{cases}$$

Thus we have

$$\begin{aligned} \langle A\mathbf{y}, \mathbf{y} \rangle &= \sum_{i=1}^{2k} \sum_{j=1}^{2k} a_{ij} y_i y_j = \sum_{i=1}^k \sum_{j=1}^k \frac{a_{ij}}{\sqrt{ij}} + \sum_{i=k+1}^{2k} \sum_{j=k+1}^{2k} \frac{a_{ij}}{\sqrt{(i-k)(j-k)}} \\ &\quad - \sum_{i=1}^k \sum_{j=k+1}^{2k} \frac{a_{ij}}{\sqrt{i(j-k)}} - \sum_{i=k+1}^{2k} \sum_{j=1}^k \frac{a_{ij}}{\sqrt{(i-k)j}} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^k \sum_{j=1}^k \frac{1}{\sqrt{ij}} \left(1 + \frac{1}{\sqrt{ij}}\right) + \sum_{i=1}^k \sum_{j=1}^k \frac{1}{\sqrt{ij}} \left(1 + \frac{1}{\sqrt{ij}}\right) \\
&\quad - \sum_{i=1}^k \sum_{j=1}^k \frac{1}{\sqrt{ij}} \left(1 - \frac{1}{\sqrt{ij}}\right) - \sum_{i=1}^k \sum_{j=1}^k \frac{1}{\sqrt{ij}} \left(1 - \frac{1}{\sqrt{ij}}\right) \\
&= 4 \sum_{i=1}^k \sum_{j=1}^k \frac{1}{\sqrt{ij}} \frac{1}{\sqrt{ij}} = 4(\zeta_k)^2.
\end{aligned}$$

Hence,

$$\mu_2(A) \geq \frac{\langle A\mathbf{y}, \mathbf{y} \rangle}{\|\mathbf{y}\|^2} \geq 2\zeta_k > 2 \log n.$$

Let us now turn to our proof of (13). Since the sum of every row of A is exactly n , we have $\rho'(A) = 1$.

Assume $X_0, Y_0 \subset [n]$ are nonempty sets, maximizing the right-hand side of (3), i.e. satisfying

$$\text{disc}(A) = \frac{1}{\sqrt{|X_0||Y_0|}} \left| \sum_{i \in X_0} \sum_{j \in Y_0} (a_{ij} - 1) \right|. \quad (14)$$

Set

$$X_1 = X_0 \cap [k], \quad X_2 = X_0 \cap [k+1, n],$$

$$Y_1 = Y_0 \cap [k], \quad Y_2 = Y_0 \cap [k+1, n].$$

Then the right-hand side of (14) is equal to

$$\begin{aligned}
&\frac{1}{\sqrt{|X_0||Y_0|}} \left| \sum_{i \in X_1} \sum_{j \in Y_1} \frac{1}{\sqrt{ij}} + \sum_{i \in X_2} \sum_{j \in Y_2} \frac{1}{\sqrt{ij}} - \sum_{i \in X_1} \sum_{j \in Y_2} \frac{1}{\sqrt{ij}} - \sum_{i \in X_2} \sum_{j \in Y_1} \frac{1}{\sqrt{ij}} \right| \\
&= \frac{1}{\sqrt{|X_0||Y_0|}} \left| \left(\sum_{i \in X_1} \frac{1}{\sqrt{i}} - \sum_{i \in X_2} \frac{1}{\sqrt{i}} \right) \left(\sum_{j \in Y_1} \frac{1}{\sqrt{j}} - \sum_{j \in Y_2} \frac{1}{\sqrt{j}} \right) \right|.
\end{aligned}$$

Since $\text{disc}(A)$ is maximal, one of X_1, X_2 is empty, and one of Y_1, Y_2 is empty. By symmetry we can assume that $X_2 = \emptyset$, $Y_2 = \emptyset$. Then the matrix $A[X_0, Y_0] = A[X_1, Y_1]$ is in the upper-left-hand corner of A and

$$\begin{aligned}
\frac{1}{\sqrt{|X_0||Y_0|}} \left| \sum_{i \in X_0} \sum_{j \in Y_0} (a_{ij} - 1) \right| &= \frac{1}{\sqrt{|X_0||Y_0|}} \left(\sum_{i=1}^{|X_0|} \frac{1}{\sqrt{i}} \right) \left(\sum_{j=1}^{|Y_0|} \frac{1}{\sqrt{j}} \right) \\
&< \frac{4\sqrt{|X_0|}\sqrt{|Y_0|}}{\sqrt{|X_0||Y_0|}} = 4. \quad \square
\end{aligned}$$

It is not impossible that the constant 4 appearing in (13) is fairly close to be the best possible.

3. A class of dense regular graphs

Our goal in this section is to construct infinitely many regular graphs G such that

$$\mu_2(G) \geq C \text{disc}_2(G) \log(|G|)$$

for some absolute constant $C > 0$. In fact, for every sufficiently large prime p and $k = \lceil p^{1/5} \rceil$ we shall construct a matrix \mathcal{A} such that:

- (a) \mathcal{A} is a square, symmetric, $(0, 1)$ -matrix of size $2kp$ with zero main diagonal;
- (b) all row sums of \mathcal{A} are equal to kp ;

(c) $\mu_2(\mathcal{A})$ satisfies

$$\mu_2(\mathcal{A}) \geq \frac{1}{2} p \log k;$$

(d) $\text{disc}(\mathcal{A})$ satisfies

$$\text{disc}(\mathcal{A}) \leq 12p.$$

The matrix \mathcal{A} will be constructed as a block matrix of $4k^2$ blocks, each block being a square matrix of size p .

We shall select a symmetric matrix of integers that is roughly proportional to the matrix A of Section 2.2, and then we shall replace each entry of that matrix by a $p \times p$, symmetric, $(0, 1)$ -matrix of low discrepancy and density equal to the value of the corresponding entry.

Before describing the blocks of \mathcal{A} , we shall consider a corollary of a theorem of Thomason.

3.1. A theorem of Thomason

Thomason [13, Theorem 2] proved a widely-applicable result about bipartite graphs with vertex classes of equal size; for convenience, we shall restate his theorem in matrix form.

Theorem 6. Let $0 < p < 1$, $\mu \geq 0$, and A be a square $(0, 1)$ -matrix of size n . If each row of A has at least pn ones, and the inner product of every two distinct rows is at most $p^2n + \mu$, then for every $X, Y \subset [n]$,

$$\left| \sum_{i \in X} \sum_{j \in Y} (a_{ij} - p) \right| \leq \varepsilon |Y| + \sqrt{|X||Y|(pn + \mu|X|)},$$

where $\varepsilon = 1$ if $p|X| < 1$ and $\varepsilon = 0$ otherwise.

Applying this theorem to the adjacency matrix of a graph G , we obtain immediately the following generalization of Theorem 1 in [13].

Theorem 7. Let $0 < p < 1$, $\mu \geq 0$, and G be a graph of order n . If $d(u) \geq pn$ for every $u \in V(G)$, and

$$|\Gamma(u) \cap \Gamma(v)| \leq p^2n + \mu$$

for every two distinct $u, v \in V(G)$, then for every $X, Y \subset V(G)$,

$$|e(X, Y) - p|X||Y|| \leq \varepsilon |Y| + \sqrt{|X||Y|(pn + \mu|X|)},$$

where $\varepsilon = 1$ if $p|X| < 1$ and $\varepsilon = 0$ otherwise.

Next we shall describe a family of symmetric $(0, 1)$ -matrices of size p that we shall use as blocks of \mathcal{A} .

3.2. The blocks of \mathcal{A}

Let p be a sufficiently large prime, \mathbb{Z}_p be the field of order p , and $t \in [p]$. Let $Q(p, t)$ be the graph whose vertex set is $[p]$, and two distinct $u, v \in [p]$ are joined if

$$\left\{ \frac{(u-v)^2}{p} \right\} \leq \frac{t}{p},$$

where $\{x\}$ is the fractional part of x . The graphs $Q(p, t)$ were introduced by Bollobás and Erdős [3], as examples of pseudo-random graphs. The following lemma summarizes the properties of $Q(p, t)$ that we shall be interested in.

Lemma 8. The graph $Q(p, t)$ is a regular graph of order p such that

(i) the degree d of $Q(p, t)$ satisfies

$$|d - t| \leq \sqrt{p}(\log p)^2;$$

(ii) the adjacency matrix A of $Q(p, t)$ satisfies

$$\text{disc}(A) < 2p^{3/4} \log p.$$

Proof. Since $Q(p, t)$ is invariant under the cyclic shift $z \rightarrow z + 1 \pmod p$, it is clear that $Q(p, t)$ is regular. In fact, (i) follows from a much stronger result of Burgess [4].

To prove (ii) we shall first recall that Theorem 3.16 in [2] states that, for any two vertices u, v of $Q(p, t)$, we have

$$\left| |\Gamma(u) \cap \Gamma(v)| - \frac{t^2}{p} \right| < \sqrt{p}(\log p)^2. \quad (15)$$

Setting $\beta = d/p$, from (i) and (15), for every two vertices u, v of $Q(p, t)$, we obtain

$$\begin{aligned} |\Gamma(u) \cap \Gamma(v)| &\leq \frac{t^2}{p} + \sqrt{p}(\log p)^2 \leq \frac{(\beta p + \sqrt{p}(\log p)^2)^2}{p} + \sqrt{p}(\log p)^2 \\ &= \beta^2 p + 2\beta\sqrt{p}(\log p)^2 + (\log p)^4 + \sqrt{p}(\log p)^2 \\ &< \beta^2 p + 3\sqrt{p}(\log p)^2 + (\log p)^4. \end{aligned}$$

Suppose that $X, Y \subset [p]$ are nonempty sets. Assuming $|X| \leq |Y|$, by Theorem 7, we obtain

$$|e(X, Y) - \beta|X||Y|| \leq |X| + \sqrt{|X||Y|} \sqrt{\beta p + (3\sqrt{p}(\log p)^2 + (\log p)^4)|Y|}.$$

Hence, noting that $|Y| \leq p$ and $\beta < 1$, we find that

$$\begin{aligned} \frac{1}{\sqrt{|X||Y|}} |e(X, Y) - \beta|X||Y|| &\leq 1 + \sqrt{\beta p + (3\sqrt{p}(\log p)^2 + (\log p)^4)p} \\ &< 2p^{3/4} \log p. \end{aligned}$$

Let $A = (a_{ij})_{i,j=1}^p$. Since, for every $X, Y \subset [p]$, we have

$$\sum_{i \in X} \sum_{j \in Y} a_{ij} = e(X, Y)$$

and $\rho'(A) = \beta$, we deduce

$$\text{disc}(A) < 2p^{3/4} \log p,$$

as claimed. \square

Let \mathcal{V}_p be the set of the degrees of the graphs $Q(p, t)$ for $t \in [p]$. From Lemma 8, (i), we see that for every $s \in [p]$ there is a $d \in \mathcal{V}_p$, such that there exists a d -regular graph $H(p, d)$ with

$$|d - s| \leq \sqrt{p} \log^2 p$$

and

$$\text{disc}_2(H(p, d)) < 2p^{3/4} \log p.$$

Now, for every $d \in \mathcal{V}_p$, let $A(p, d)$ be the adjacency matrix of $H(p, d)$. The properties of the matrices $\{A(p, d) : d \in \mathcal{V}_p\}$ are summarized in the following lemma.

Lemma 9. For every integer $s \in [p]$, there exist $d \in \mathcal{V}_p$ and a matrix $A(p, d)$, such that

- (i) $|d - s| < \sqrt{p} \log^2 p$;
- (ii) $A(p, d)$ is a symmetric $(0, 1)$ -matrix of size p with zero main diagonal;
- (iii) all row sums of $A(p, d)$ are equal to d ;
- (iv) the function $\text{disc}(A(p, d))$ satisfies

$$\text{disc}(A(p, d)) < 2p^{3/4} \log p.$$

If A is a square $(0, 1)$ -matrix of size n , we call the matrix

$$\bar{A} = E_n - A$$

the complement of A . Observe that if A is a square $(0, 1)$ -matrix then

$$\rho'(\bar{A}) = 1 - \rho'(A),$$

$$\text{disc}(\bar{A}) = \text{disc}(A).$$

Hence, the complement of any matrix $A(p, d)$ satisfies

$$\text{disc}(\bar{A}(p, d)) < 2p^{3/4} \log p.$$

The matrices $\{A(p, d) : d \in \mathcal{V}_p\}$ together with their complements will be used as blocks of the matrix \mathcal{A} .

3.3. The construction of \mathcal{A}

For every $s \in [2k]$, set

$$I_s = \{i : (s-1)p \leq i < sp\}.$$

Define the matrix $D = (d_{ij})_{i,j=1}^k$ by

$$d_{ij} = q, \quad q \in \mathcal{V}_p, \quad \left| q - \left(\frac{p}{2} + \frac{p}{2\sqrt{ij}} \right) \right| = \min_{x \in \mathcal{V}_p} \left| x - \left(\frac{p}{2} + \frac{p}{2\sqrt{ij}} \right) \right|.$$

From Lemma 8, (i), we see that

$$\left| 2d_{ij} - \left(p + \frac{p}{\sqrt{ij}} \right) \right| \leq 2\sqrt{p} \log^2 p. \quad (16)$$

The matrix D will be the cornerstone of our construction. Note that D is symmetric and the values of its entries belong to the set \mathcal{V}_p .

Now, let us define \mathcal{A}' as a block matrix by

$$\mathcal{A}' = \begin{pmatrix} A(p, d_{11}) & A(p, d_{12}) & \cdots & A(p, d_{1k}) \\ A(p, d_{12}) & A(p, d_{22}) & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A(p, d_{1k}) & \cdot & \cdot & A(p, d_{kk}) \end{pmatrix} \quad (17)$$

and set

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}' & E_{kp} - \mathcal{A}' \\ E_{kp} - \mathcal{A}' & \mathcal{A}' \end{pmatrix}. \quad (18)$$

By our construction \mathcal{A} is a symmetric $(0,1)$ -matrix of size $2pk$, and its main diagonal is zero, so \mathcal{A} satisfies (a). Also, we see that every row sum of \mathcal{A} is exactly kp , so \mathcal{A} satisfies (b) as well. In the following two theorems we shall prove that \mathcal{A} satisfies also (c) and (d).

For the sake of convenience, set $\mathcal{A} = (a_{ij})_{i,j=1}^{2pk}$ and $\mathcal{A}_{ij} = \mathcal{A}[I_i, I_j]$ for $i, j \in [2k]$. Observe that the row sums of any matrix \mathcal{A}_{ij} are equal, and from Lemma 9 and what follows, we have

$$\text{disc}(\mathcal{A}_{ij}) \leq 2p^{3/4} \log p. \quad (19)$$

Theorem 10. *The second eigenvalue $\mu_2(\mathcal{A})$ of the matrix \mathcal{A} defined by (18) satisfies*

$$\mu_2(\mathcal{A}) \geq \frac{1}{2} p \log k.$$

Proof. Indeed, from (18) we see that the sum of every row of \mathcal{A} is exactly kp . Since \mathcal{A} is nonnegative, it follows that $\mu_1(\mathcal{A}) = pk$ and the vector $\mathbf{j} \in \mathbb{R}^{2pk}$ of all ones is an eigenvector of \mathcal{A} to $\mu_1(\mathcal{A})$. By the Rayleigh principle

$$\mu_2(\mathcal{A}) = \max_{\mathbf{y} \perp \mathbf{j}, \mathbf{y} \neq \mathbf{0}} \frac{\langle \mathcal{A}\mathbf{y}, \mathbf{y} \rangle}{\|\mathbf{y}\|^2},$$

so our goal is to find a nonzero $\mathbf{y} \in \mathbb{R}^{2pk}$ such that $\mathbf{y} \perp \mathbf{j}$ and the ratio $\langle \mathcal{A}\mathbf{y}, \mathbf{y} \rangle / \|\mathbf{y}\|^2$ is sufficiently large.

Define the vector $\mathbf{y} = (y_i)_{i=1}^{2pk}$ by

$$y_i = \begin{cases} 1/\sqrt{s} & \text{if } i \in I_s, \quad s \leq k, \\ -1/\sqrt{s-k} & \text{if } i \in I_s, \quad s > k. \end{cases}$$

From

$$\sum_{i=1}^{2pk} y_i = \sum_{s=1}^{2k} \sum_{i \in I_s} \frac{1}{\sqrt{s}} - \sum_{s=k+1}^{2k} \sum_{i \in I_s} \frac{1}{\sqrt{s-k}} = \sum_{s=1}^k \frac{p}{\sqrt{s}} - \sum_{s=1}^k \frac{p}{\sqrt{s}} = 0,$$

we see that $\mathbf{y} \perp \mathbf{j}$. Also, for $\|\mathbf{y}\|^2$ we have

$$\|\mathbf{y}\|^2 = \sum_{s=1}^k \sum_{i \in I_s} \frac{1}{s} + \sum_{s=k+1}^{2k} \sum_{i \in I_s} \frac{1}{s-k} = 2 \sum_{s=1}^k \sum_{i \in I_s} \frac{1}{s} = 2p \sum_{s=1}^k \frac{1}{s} = 2p \zeta_k.$$

On the other hand, for $\langle \mathcal{A}\mathbf{y}, \mathbf{y} \rangle$ we see that

$$\langle \mathcal{A}\mathbf{y}, \mathbf{y} \rangle = \sum_{i=1}^{2pk} \sum_{j=1}^{2pk} a_{ij} y_i y_j = \sum_{i=1}^{2k} \sum_{j=1}^{2k} \sum_{s \in I_i} \sum_{t \in I_j} a_{st} y_s y_t.$$

By (17) and (18), we have

$$\sum_{s \in I_i} \sum_{t \in I_j} a_{st} = \begin{cases} pd_{ij} & \text{if } i \leq k, j \leq k, \\ p(p - d_{i(j-k)}) & \text{if } i \leq k, j > k, \\ p(p - d_{(i-k)j}) & \text{if } i > k, j \leq k, \\ pd_{(i-k)(j-k)} & \text{if } i > k, j > k. \end{cases}$$

Hence, by the choice of \mathbf{y} ,

$$\begin{aligned} \langle \mathcal{A}\mathbf{y}, \mathbf{y} \rangle &= \sum_{i=1}^k \sum_{j=1}^k \frac{pd_{ij}}{\sqrt{ij}} + \sum_{i=k+1}^{2k} \sum_{j=k+1}^{2k} \frac{pd_{(i-k)(j-k)}}{\sqrt{(i-k)(j-k)}} \\ &\quad - \sum_{i=1}^k \sum_{j=k+1}^{2k} \frac{p(p - d_{i(j-k)})}{\sqrt{i(j-k)}} - \sum_{i=k+1}^{2k} \sum_{j=1}^k \frac{p(p - d_{(i-k)j})}{\sqrt{(i-k)j}} \\ &= 2p \left(\sum_{i=1}^k \sum_{j=1}^k \frac{d_{ij}}{\sqrt{ij}} - \sum_{i=1}^k \sum_{j=1}^k \frac{p - d_{ij}}{\sqrt{ij}} \right) = 2p \left(\sum_{i=1}^k \sum_{j=1}^k \frac{2d_{ij} - p}{\sqrt{ij}} \right). \end{aligned}$$

From (16), we have

$$\frac{2d_{ij} - p}{\sqrt{ij}} > \frac{1}{\sqrt{ij}} \left(\frac{p}{\sqrt{ij}} - 2\sqrt{p}(\log p)^2 \right) = \frac{p}{ij} - \frac{2\sqrt{p}(\log p)^2}{\sqrt{ij}}$$

and so,

$$\begin{aligned} \langle \mathcal{A}\mathbf{y}, \mathbf{y} \rangle &> 2p^2 \left(\sum_{i=1}^k \sum_{j=1}^k \frac{1}{ij} \right) - 4p\sqrt{p}(\log p)^2 \sum_{i=1}^k \sum_{j=1}^k \frac{1}{\sqrt{ij}} \\ &> 2p^2(\zeta_k)^2 - 16kp\sqrt{p}(\log p)^2. \end{aligned}$$

Hence, as $k \leq p^{1/5}$ and p is large, $\langle \mathcal{A}\mathbf{y}, \mathbf{y} \rangle > p^2(\zeta_k)^2$, and thus,

$$\mu_2(\mathcal{A}) \geq \frac{\langle \mathcal{A}\mathbf{y}, \mathbf{y} \rangle}{\|\mathbf{y}\|^2} \geq \frac{1}{2} p \zeta_k > \frac{1}{2} p \log k$$

as claimed. \square

Theorem 11. If p is large, $\text{disc}(\mathcal{A})$ of the matrix \mathcal{A} defined by (18) satisfies

$$\text{disc}(\mathcal{A}) \leq 12p.$$

Proof. Since all row sums of \mathcal{A} are exactly pk , we deduce $\rho'(\mathcal{A}) = \frac{1}{2}$.

As before, assume $X_0, Y_0 \subset [2kp]$ are nonempty sets, maximizing the right-hand side of (3), i.e. satisfying

$$\text{disc}(\mathcal{A}) = \frac{1}{\sqrt{|X_0||Y_0|}} \left| \sum_{i \in X_0} \sum_{j \in Y_0} \left(a_{ij} - \frac{1}{2} \right) \right|. \quad (20)$$

Set

$$J_1 = [kp], \quad J_2 = [kp + 1, 2kp]$$

and let

$$X_i = X_0 \cap J_i, \quad Y_i = Y_0 \cap J_i, \quad i = 1, 2.$$

For $i, j = 1, 2$ consider the value

$$\Delta_{ij} = \max_{X \subset J_i, Y \subset J_j, X \neq \emptyset, Y \neq \emptyset} \frac{1}{\sqrt{|X||Y|}} \left| \sum_{i \in X} \sum_{j \in Y} \left(a_{ij} - \frac{1}{2} \right) \right|.$$

By (18), we have

$$\mathcal{A}[J_1, J_1] = \mathcal{A}[J_2, J_2],$$

$$\mathcal{A}[J_1, J_2] = \mathcal{A}[J_2, J_1] = E_{kn} - \mathcal{A}[J_1, J_1]$$

and hence,

$$A_{11} = A_{12} = A_{21} = A_{22}.$$

Consequently

$$\begin{aligned} \text{disc}(\mathcal{A}) &= \frac{1}{\sqrt{|X_0||Y_0|}} \left| \sum_{i=1}^2 \sum_{j=1}^2 \sum_{s \in X_i} \sum_{t \in Y_j} \left(a_{st} - \frac{1}{2} \right) \right| \\ &\leq \frac{1}{\sqrt{|X_0||Y_0|}} \sum_{i=1}^2 \sum_{j=1}^2 A_{ij} \sqrt{|X_i||Y_j|} \\ &= \frac{A_{11}}{\sqrt{|X_0||Y_0|}} \left(\sqrt{|X_1|} + \sqrt{|X_2|} \right) \left(\sqrt{|Y_1|} + \sqrt{|Y_2|} \right) \\ &\leq 2A_{11} \frac{\sqrt{(|X_1| + |X_2|)(|Y_1| + |Y_2|)}}{\sqrt{(|X_1| + |X_2|)(|Y_1| + |Y_2|)}} = 2A_{11}. \end{aligned}$$

To complete our proof we shall show that

$$A_{11} < 6p.$$

Fix some nonempty sets $X_0, Y_0 \subset [kp]$ such that

$$A_{11} = \frac{1}{\sqrt{|X_0||Y_0|}} \left| \sum_{i=1}^k \sum_{j=1}^k \sum_{s \in X_i} \sum_{t \in Y_j} \left(a_{st} - \frac{1}{2} \right) \right| \quad (21)$$

and for every $i \in [k]$, set

$$X_i = X_0 \cap I_i, \quad Y_i = Y_0 \cap I_i.$$

Observe that for $i, j \in [k]$ we have

$$\rho'(\mathcal{A}_{ij}) - \frac{1}{2} = \frac{d_{ij}}{p} - \frac{1}{2},$$

hence, by (16),

$$\left| \rho'(\mathcal{A}_{ij}) - \frac{1}{2} \right| < \frac{1}{2\sqrt{ij}} + \frac{2(\log p)^2}{\sqrt{p}}$$

and so,

$$\begin{aligned} \left| \sum_{s \in X_i} \sum_{t \in Y_j} \left(a_{st} - \frac{1}{2} \right) \right| &\leq \left| \sum_{s \in X_i} \sum_{t \in Y_j} (a_{st} - \rho'(\mathcal{A}_{ij})) \right| + |X_i||Y_j| \left| \rho'(\mathcal{A}_{ij}) - \frac{1}{2} \right| \\ &\leq \text{disc}(\mathcal{A}_{ij}) \sqrt{|X_i||Y_j|} + |X_i||Y_j| \left(\frac{1}{2\sqrt{ij}} + \frac{2(\log p)^2}{\sqrt{p}} \right). \end{aligned}$$

Recalling (21), we see that

$$\begin{aligned} A_{11} &\leq \frac{1}{\sqrt{|X_0||Y_0|}} \sum_{i=1}^k \sum_{j=1}^k \left| \sum_{s \in X_i} \sum_{t \in Y_j} \left(a_{st} - \frac{1}{2} \right) \right| \\ &\leq \frac{1}{\sqrt{|X_0||Y_0|}} \sum_{i=1}^k \sum_{j=1}^k \text{disc}(\mathcal{A}_{ij}) \sqrt{|X_i||Y_j|} \quad (22) \\ &\quad + \frac{1}{2\sqrt{|X_0||Y_0|}} \sum_{i=1}^k \sum_{j=1}^k \frac{|X_i||Y_j|}{\sqrt{ij}} \quad (23) \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2(\log p)^2}{\sqrt{p}} \right) \frac{1}{\sqrt{|X_0||Y_0|}} \sum_{i=1}^k \sum_{j=1}^k |X_i||Y_j| \\
& = A + B + C.
\end{aligned} \tag{24}$$

We shall estimate the terms (22)–(24) separately.

From (19) we obtain

$$A \leq \frac{2p^{3/4} \log p}{\sqrt{|X_0||Y_0|}} \sum_{i=1}^k \sum_{j=1}^k \sqrt{|X_i||Y_j|}.$$

Hence, by

$$\begin{aligned}
\frac{1}{\sqrt{|X_0||Y_0|}} \sum_{i=1}^k \sum_{j=1}^k \sqrt{|X_i||Y_j|} &= \frac{1}{\sqrt{|X_0|}} \left(\sum_{i=1}^k \sqrt{|X_i|} \right) \frac{1}{\sqrt{|Y_0|}} \left(\sum_{j=1}^k \sqrt{|Y_j|} \right) \\
&\leq \sqrt{k} \sqrt{k} = k,
\end{aligned}$$

we have

$$A \leq 2kp^{3/4} \log p \leq p. \tag{25}$$

Next we turn to (23). Obviously,

$$B = \frac{1}{2\sqrt{|X_0|}} \left(\sum_{i=1}^k \frac{|X_i|}{\sqrt{i}} \right) \frac{1}{\sqrt{|Y_0|}} \left(\sum_{i=1}^k \frac{|Y_i|}{\sqrt{i}} \right). \tag{26}$$

We shall show that

$$\frac{1}{\sqrt{|X_0|}} \left(\sum_{i=1}^k \frac{|X_i|}{\sqrt{i}} \right) \leq 2\sqrt{2p}. \tag{27}$$

Indeed, set

$$s = \left\lfloor \frac{|X_0|}{p} \right\rfloor$$

and observe that the left-hand side of (27) attains its maximum when

$$|X_i| = p, \quad 1 \leq i \leq s,$$

$$|X_{s+1}| = |X_0| - ps,$$

$$|X_i| = 0, \quad s+1 < i \leq 2k.$$

Obviously (27) holds if $s = 0$, so we shall assume $s \geq 1$. Then we have,

$$\frac{1}{\sqrt{|X_0|}} \left(\sum_{i=1}^k \frac{|X_i|}{\sqrt{i}} \right) \leq \frac{1}{\sqrt{|X_0|}} \sum_{i=1}^{s+1} \frac{p}{\sqrt{i}} \leq \frac{2p\sqrt{s+1}}{\sqrt{|X_0|}} \leq \sqrt{2p \frac{sp}{|X_0|}} \leq 2\sqrt{2p}$$

and (27) follows.

Similarly, we see that

$$\frac{1}{\sqrt{|Y_0|}} \left(\sum_{i=1}^k \frac{|Y_i|}{\sqrt{i}} \right) \leq 2\sqrt{2p}$$

and hence, in view of (26), we find

$$B \leq 4p. \tag{28}$$

Finally,

$$C = \left(\frac{2(\log p)^2}{\sqrt{p}} \right) \frac{|X||Y|}{\sqrt{|X||Y|}} \leq \left(\frac{2(\log p)^2}{\sqrt{p}} \right) \sqrt{kp} < p. \tag{29}$$

Now, replacing (22)–(24) by (25), (28), (29), we obtain

$$A_{11} < 6p,$$

and the proof is completed. \square

3.4. A conjecture of Chung

In [6] Chung studies a version of the Laplacian matrix a graph G that she denotes by $\mathcal{L}(G)$. If G is d -regular of order n the matrix $\mathcal{L}(G)$ is given by

$$\mathcal{L}(G) = I_n - \frac{1}{d} A, \quad (30)$$

where A is the adjacency matrix of $G(n)$. Following Chung's notation, the eigenvalues of $\mathcal{L}(G)$ are $\lambda_0 \leq \dots \leq \lambda_{n-1}$, with $\lambda_0 = 0$.

Set $\tilde{\lambda} = \max_{i \neq 0} |1 - \lambda_i|$, and for every $X \subset V(G)$, let $\text{vol } X = \sum_{v \in X} d(v)$. Chung asked the following question.

Let G be a nonempty graph and $\alpha > 0$ is such that if $X, Y \subset V = V(G)$ then

$$\left| e(X, Y) - \frac{\text{vol } X \text{ vol } Y}{\text{vol } V} \right| \leq \alpha \frac{\sqrt{\text{vol } X \text{ vol } Y \text{ vol}(V \setminus X) \text{ vol}(V \setminus Y)}}{\text{vol } V}. \quad (31)$$

Is there an absolute constant C such that $\tilde{\lambda} \leq C\alpha$?

We shall check that the graph G_p , whose adjacency matrix $\mathcal{A}_p = \mathcal{A}$ we constructed in the previous section, answers this question in the negative. Indeed, recall that G_p is kp -regular graph of order $n = 2kp$. Theorem 11 implies that

$$\left| e(X, Y) - \frac{|X||Y|}{2} \right| \leq \frac{C}{k} \sqrt{|X||Y|(n - |X|)(n - |Y|)}$$

for some absolute constant $C > 0$, so (31) holds with $\alpha = C/k$. By (30) and Theorem 10, we see that

$$\tilde{\lambda} \geq |1 - \lambda_1| \geq 1 - \left(1 - \frac{\mu_2}{\mu_1}\right) = \frac{\mu_2}{\mu_1} \geq \frac{p \log k}{2kp} = \frac{\log k}{2k}$$

and $\tilde{\lambda}$ is greater than any fixed multiple of α .

4. Sparse graphs with low discrepancy and high second eigenvalue

In [7] Chung and Graham extend quasi-random properties to sparse graphs, i.e., graphs $G(n, m)$ with $m = o(n^2)$. Their approach is based on the following. Fix a function $p = p(n)$ with $0 < p < 1$ and

$$\lim_{n \rightarrow \infty} pn = \infty.$$

Let \mathcal{G}_p be an infinite family of graphs $\{G(n) : n \rightarrow \infty\}$ such that, for every $G(n) \in \mathcal{G}_p$,

$$e(G(n)) = (1 + o(1))p \binom{n}{2}. \quad (32)$$

Chung and Graham investigated a number of properties that a family \mathcal{G}_p can have; we shall be concerned with the following two here [7, p. 220]:

DISC(1): For every $G(n) \in \mathcal{G}_p$, and for all $X, Y \subset V(G)$,

$$|e(X, Y) - p|X||Y|| = o(pn^2).$$

EIG: For every $G(n) \in \mathcal{G}_p$,

$$\mu_1(G) = (1 + o(1))pn \quad \text{and} \quad \sigma_2(G) = o(pn).$$

Chung and Graham proved that EIG implies DISC(1) [7, Theorem 1], and asked the following natural question [7, p. 230]

Question: Does DISC(1) imply EIG?

Recently Krivelevich and Sudakov [10, p. 9] constructed an example that answers this question in the negative. To conclude the paper we give a general construction that we believe sheds more light on the relationship between DISC(1) and EIG.

Proposition 12. For $p = p(n) = o(1)$ let \mathcal{G}_p be a family of graphs having the property EIG. Let \mathcal{G}_p^* be the family of the graphs that can be represented as disjoint unions

$$G(n) \cup K_{\lfloor pn \rfloor},$$

where $G(n) \in \mathcal{G}_p$. Then \mathcal{G}_p^* has DISC(1) but does not have EIG.

Proof. Note that

$$e(G(n) \cup K_{\lfloor pn \rfloor}) = (1 + o(1))p \binom{n}{2} + \binom{\lfloor pn \rfloor}{2} = (1 + o(1))p \binom{n + \lfloor pn \rfloor}{2},$$

so \mathcal{G}_p^* is defined according to (32). Also, given $G' = G(n) \cup K_{\lfloor pn \rfloor}$, $Z = V(K_{\lfloor pn \rfloor})$ and $X, Y \subset V(G')$, we have

$$\begin{aligned} |e(X, Y) - p|X||Y|| &\leq |e(X \setminus Z, Y \setminus Z) - p|X \setminus Z||Y \setminus Z|| \\ &\quad + |e(X, Y) - e(X \setminus Z, Y \setminus Z)| + |p|X||Y| - p|X \setminus Z||Y \setminus Z|| \\ &\leq o(pn^2) + 2e(Z) + p|Z|(|X| + |Y|) \\ &\leq o(pn^2) + p^2n^2 + 2p^2n^2 = o(pn^2). \end{aligned}$$

Thus, \mathcal{G}_p^* has DISC(1). However, since G' is a union of the disjoint graphs $G(n)$ and $K_{\lfloor pn \rfloor}$, we find that

$$\begin{aligned} \min\{\mu_1(G(n)), \mu_1(K_{\lfloor pn \rfloor})\} &\leq \mu_2(G') \leq \mu_1(G'(n)) \\ &= \max\{\mu_1(G(n)), \mu_1(K_{\lfloor pn \rfloor})\}. \end{aligned}$$

Hence, from $\mu_1(G(n)) = (1 + o(1))pn$ and $\mu_1(K_{\lfloor pn \rfloor}) = \lfloor pn \rfloor - 1$, we see that

$$\mu_2(G') = (1 + o(1))pn,$$

and so, \mathcal{G}_p^* does not have EIG. \square

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